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Compact spaces representable as unions of nice subspaces

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Abstract

We present several results on compact Hausdorff spaces which can be represented as unions of nice subspaces. Some typical results are: If X is a compact Hausdorff space, and $X = \bigcup_{\alpha<\kappa} X_\alpha$, where each X_α is κ-refinable and $\Psi(X_\alpha) \leqslant \kappa$, then (i) every nonempty G_κ-subset of X contains a point of character $\leqslant \kappa$, (ii) if $x \in X$, $\chi(x, X) = \mu > \kappa$ and μ is regular, then there exists a discrete sequence $\{x_\alpha : \alpha < \mu\}$ in X such that $x_\alpha \to x$, (iii) if A is a nonclosed subset of X, then there exists a point $x \in X \setminus A$ and a filter base $\mathscr{F}$ of subsets of A such that $|\mathscr{F}| \leqslant \kappa$ and $\mathscr{F} \to x$. We also show that if a compact Hausdorff space X is a union of countably many metrizable spaces, X has no isolated points and $c(X) = \omega_0$, then X is a compactification of the space of irrationals.

Keywords: Compact; Initially κ-compact; κ-refinable; (Pseudo)character, Filter base; (Discrete) convergent sequence

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0. Introduction

Throughout this paper, κ denotes an infinite cardinal number. All spaces are assumed to be T_1.

A topological space X is called *initially κ-compact* if every open cover of X of cardinality $\leqslant \kappa$ has a finite subcover. It is well known that a space X is initially κ-compact if and only if every infinite subset of X of cardinality $\leqslant \kappa$ has a point of complete accumulation in X (see [12]).

A space X is called *κ-refinable* if every open cover $\mathscr{G}$ of X has an open refinement $\mathscr{H}$ such that $\mathscr{H} = \bigcup_{\alpha<\kappa} \mathscr{H}_\alpha$ and for each $x \in X$ there exists $\alpha < \kappa$ such that $1 \leqslant |\{V \in \mathscr{H}_\alpha : x \in V\}| \leqslant \kappa$.

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Note that the ω_0-refinable spaces are the same as the weakly $\delta\theta$-refinable spaces introduced by Wicke and Worrell in [13].

The following theorem is known [8,13].

Theorem 1. *An initially κ-compact κ-refinable space is compact.*

In this paper, we consider the following general question: "If X is an initially κ-compact Hausdorff space and $X = \bigcup_{\alpha<\kappa} X_\alpha$, where each X_α is κ-refinable and $\psi(X_\alpha) \leqslant \kappa$, then what can be said about the space X?" Since a union of κ or less κ-refinable spaces is itself κ-refinable, in view of the above theorem, we can assume that the space X is compact. Compact spaces which are unions of certain collections of subspaces of special type were first considered by Arhangel'skiĭ in [2,3]. Most results in [2,3] pertain to the situation where each subspace X_α is either metrizable or, more generally, belongs to a certain class $\mathscr{E}_\kappa$ of spaces. The class of spaces which are κ-refinable and have pseudocharacter $\leqslant \kappa$ and the class $\mathscr{E}_\kappa$ both contain the class of metrizable spaces but are, in general, incomparable.

We adopt the usual notation for cardinal functions considered here. In particular, $\psi(x, X)$ is the psuedocharacter of a point x in X and $\chi(x, X)$ is the character of x in X. Also $\psi(X)$, $\chi(X)$ and $t(X)$ denote, respectively, the pseudocharacter, the character and the tightness of X. Furthermore, $\omega(X)$, $\pi\omega(X)$, $d(X)$ and $c(X)$ denote, respectively, the weight of X, the π-weight of X, the density of X and the cellularity of X. Terms not defined here can be found in [4].

1.

We first prove the following lemmas.

A subset of X is called a G_κ-set if it can be represented as an intersection of κ or less open subsets of X.

Lemma 2. *Let X be a regular initially κ-compact space, and let $X = \bigcup_{\alpha<\kappa} X_\alpha$.*

(a) *If, for each $\alpha < \kappa$, $\psi(X_\alpha) \leqslant \kappa$, then every nonempty G_κ-subset of X contains a point of character $\leqslant \kappa$ in X.*

(b) *If κ is an uncountable regular cardinal, $\psi(X_\alpha) < \kappa$, for each $\alpha < \kappa$, and $\lambda < \kappa$, then every nonempty G_λ-subset of X contains a point of character $< \kappa$ in X.*

Proof. (a) Let U be a nonempty G_κ-subset of X and suppose that for each $x \in U$, $\chi(x, X) > \kappa$. By induction, we choose a decreasing sequence $\{F_\alpha\colon \alpha < \kappa\}$ of nonempty closed G_κ-subsets of X as follows:

If $U \cap X_0 = \emptyset$, let F_0 be an arbitrary nonempty closed G_κ-subset of X such that $F_0 \subseteq U$. If $U \cap X_0 \neq \emptyset$, let $x \in U \cap X_0$. Let V be a G_κ-subset of X such that $V \cap X_0 = \{x\}$. Since $\chi(x, X) > \kappa$ and X is initially κ-compact regular space, $\psi(x,$

$X) > \kappa$. Therefore, $(U \cap V)\setminus\{x\} \neq \emptyset$. Let F_0 be a nonempty closed G_κ-subset of X such that $F_0 \subseteq (U \cap V)\setminus\{x\}$. Then $F_0 \cap X_0 = \emptyset$.

If $\beta < \kappa$ and, for each $\alpha < \beta$, we have chosen F_α, then, since $\bigcap_{\alpha<\beta} F_\alpha$ is a nonempty G_κ-subset of X, by repeating the above argument with $\bigcap_{\alpha<\beta} F_\alpha$ in place of U and X_β in place of X_0, we can find a nonempty closed G_κ-subset F_β of X such that $F_\beta \subseteq \bigcap_{\alpha<\beta} F_\alpha$ and $F_\beta \cap X_\beta = \emptyset$.

Let $F = \bigcap_{\alpha<\kappa} F_\alpha$. Since X is initially κ-compact, $F \neq \emptyset$. On the other hand, since $F \cap X_\alpha = \emptyset$, for each $\alpha < \kappa$, $F = \emptyset$. This is a contradiction.

The second part of the lemma can be proved in essentially the same way. □

Corollary 3. *Let X be either*

(a) *a regular countably compact space such that $X = \bigcup_{i=1}^{\infty} X_i$, where $\psi(X_i) \leqslant \omega_0$, for each i, or*

(b) *a regular initially ω_1-compact space such that $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\psi(X_\alpha) \leqslant \omega_0$, for each $\alpha < \omega_1$.*

Let $Y = \{x \in X\colon \chi(x, X) \leqslant \omega_0\}$. Then for every nonempty open subset U of X, $U \cap Y$ is of second category in X.

Proof. Assume the contrary. Then by the Baire Category Theorem, there exists a dense G_δ-subset G of X such that $(U \cap Y) \cap G = \emptyset$. In view of the above lemma, this is a contradiction since $U \cap G$ is a nonempty G_δ-subset of X. □

Remark 4. Let X be a regular space, and let $X = \bigcup_{\alpha<\kappa} X_\alpha$, where $\chi(X_\alpha) \leqslant \kappa$, for each $\alpha < \kappa$. Let $Y' = \{x \in X\colon \chi(x, X) > \kappa\}$. Then, for each $\alpha < \kappa$, $Y' \cap X_\alpha$ is nowhere dense in X. Thus $Y' = \bigcup_{\alpha<\kappa}(Y' \cap X_\alpha)$ is a union of κ or less nowhere dense subsets of X. If X is assumed to be initially κ-compact (or compact), can Y' be dense in X? See a related problem, Problem 18 below.

The following lemma, for $\kappa = \omega_0$, follows from [13, Theorem 1.1]. By slightly modifying the proof of [8, Theorem 1], one can also obtain a short and elementary proof of this lemma, which we present below.

Lemma 5. *If X is a κ-refinable space and Y is an initially κ-compact subspace of X, then every open cover of X contains a finite subfamily which covers Y.*

Proof. Assume the contrary, and let $\mathscr{G}$ be a maximal open cover of X no finite subfamily of which covers Y. Let $\mathscr{H} = \bigcup_{\alpha<\kappa} \mathscr{H}_\alpha$ be an open refinement of $\mathscr{G}$ which witnesses the κ-refinability of X. For each $\alpha < \kappa$, and for each $x \in X$, let $\mathscr{H}_\alpha(x) = \{V \in \mathscr{H}_\alpha\colon\ x \in V\}$ and $X_\alpha = \{x \in X\colon\ 1 \leqslant |\mathscr{H}_\alpha(x)| \leqslant \kappa\}$. Then $X = \bigcup_{\alpha<\kappa} X_\alpha$. Since Y is initially κ-compact, there exists β such that $Y \cap X_\beta$ cannot be covered by κ or less members of $\mathscr{G}$. Let $W = \bigcup \mathscr{H}_\beta$. Since $X_\beta \subseteq W$, $W \notin \mathscr{G}$. By the maximality of $\mathscr{G}$, there exists $U \in \mathscr{G}$ such that $Y \subseteq W \cup U$. Then $(Y \cap X_\beta)\setminus U$ cannot be covered by κ or less members of $\mathscr{G}$.

By induction, we choose a sequence $y_1, y_2, \ldots$ of points in $(Y \cap X_\beta) \setminus U$ as follows: Let $y_1 \in (Y \cap X_\beta) \setminus U$ be arbitrary. If $y_1, \ldots, y_n$ have already been chosen, then, since $|\bigcup_{i=1}^n \mathscr{H}_\beta(y_i)| \leqslant \kappa$, $(Y \cap X_\beta) \setminus U$ is not contained in $\bigcup(\bigcup_{i=1}^n \mathscr{H}_\beta(y_i))$. Choose $y_{n+1} \in ((Y \cap X_\beta) \setminus U) \setminus \bigcup(\bigcup_{i=1}^n \mathscr{H}_\beta(y_i))$.

Let $S = \{y_1, y_2, \ldots\}$. Then $S \subseteq Y \setminus U$ and, since $Y \setminus U \subseteq W$, no point of $Y \setminus U$ is a limit point of S. This is a contradiction, since $Y \setminus U$ is initially κ-compact. □

If $\lambda \geqslant \omega_0$, then we say that a sequence $\{x_\alpha: \alpha < \lambda\}$ in X converges to a set $C \subseteq X$ ($x_\alpha \to C$), if for every open set U containing C, there exists $\beta < \lambda$ such that $x_\alpha \in U$, for each $\alpha \geqslant \beta$.

Theorem 6. (*Main*). *Let X be a compact Hausdorff space, and let $X = \bigcup_{\alpha<\kappa} X_\alpha$, where each X_α is κ-refinable and $\psi(X_\alpha) \leqslant \kappa$. Then the following hold.*

(1) *Let $Y = \{x \in X: \chi(x, X) \leqslant \kappa\}$. Then Y is dense in X. Also, if a collection $\{B_\alpha: \alpha < \kappa\}$ of subsets of X covers Y, then $\bigcup\{\overline{B}_\alpha: \alpha < \kappa\} = X$.*

(2) *Initially κ-compact subsets of X are closed in X.*

(3) *If A is a nonclosed subset of X, then there exist a nonempty compact set $C \subseteq X \setminus A$ and a sequence $\{x_\alpha: \alpha < \lambda\} \subseteq A$, where λ is a regular cardinal $\leqslant \kappa$, such that $x_\alpha \to C$.*

(4) *If A is a nonclosed subset of X, then there exist a point $x \in X \setminus A$ and a filter base $\mathscr{F}$ of subsets of A such that $|\mathscr{F}| \leqslant \kappa$ and $\mathscr{F} \to x$ (i.e., each neighborhood of x contains a member of $\mathscr{F}$).*

(5) *$t(X) \leqslant \kappa$.*

(6) *There exists a dense subset D of X such that $w(D) \leqslant d(X) \cdot \kappa$.*

(7) *$|X| \leqslant 2^{c(X) \cdot \kappa}$.*

(8) *If $\chi(x, X) = \mu > \kappa$, where μ is a regular cardinal, then there exists a discrete sequence $\{x_\alpha: \alpha < \mu\}$ in X such that $x_\alpha \to x$.*

Proof. (1) This follows from Lemma 2(a).

(2) Let Z be an initially κ-compact subset of X and let $x \in X \setminus Z$. For each $y \in X$, $y \neq x$, let V_y be an open subset of X such that $y \in V_y$ and $x \notin \overline{V}_y$. Let $\mathscr{G} = \{V_y: y \in X \setminus \{x\}\}$. If $x \in X_\alpha$, then, since $\psi(x, X_\alpha) \leqslant \kappa$, $X_\alpha \setminus \{x\}$ is a union of κ or less closed subsets of X_α. Therefore, $X_\alpha \setminus \{x\}$ is κ-refinable. If $x \notin X_\alpha$, then $X_\alpha \setminus \{x\} = X_\alpha$ is again κ-refinable. Hence, $X \setminus \{x\} = \bigcup_{\alpha<\kappa}(X_\alpha \setminus \{x\})$ is κ-refinable. Therefore, by Lemma 5, there exist finitely many $V_1, \ldots, V_n \in \mathscr{G}$ such that $Z \subseteq \bigcup_{i=1}^n V_i$. Since, $x \notin \overline{V}_i$, for each i, $x \notin \overline{Z}$. This shows that Z is closed in X.

(3) We will show that (2) is, in fact, equivalent to (3) for any compact Hausdorff space X.

(2) ⇒ (3) Let A be a nonclosed subset of X. Then by (2) A is not initially κ-compact. Let λ be the smallest cardinal such that A is not initially λ-compact. Then λ is a regular cardinal (see [12]) and $\lambda \leqslant \kappa$. Let $S = \{x_\alpha: \alpha < \lambda\}$ be a sequence in A which has no point of complete accumulation in A. Let C be the

set of all points of complete accumulation of S in X. Then $C \neq \emptyset$, C is compact and $C \subseteq X \setminus A$. If U is an open set containing C, then $|S \setminus U| < \lambda$. Indeed, if $|S \setminus U| = \lambda$, and y is a point of complete accumulation of $S \setminus U$, then $y \in \overline{S \setminus U} \subseteq X \setminus U$ and $y \in C \subseteq U$. This is a contradiction. Hence, there exists $\beta < \lambda$ such that $x_\alpha \in U$, for each $\alpha \geqslant \beta$.

(3) $\Rightarrow$ (2) Let A be a nonclosed subset of X. By (3), let $S = \{x_\alpha\colon \alpha < \lambda\}$ be a sequence in A, where λ is a regular cardinal $\leqslant \kappa$, and let C be a nonempty compact subset of X such that $C \subseteq X \setminus A$ and $x_\alpha \to C$. Then any point of complete accumulation of S must belong to C. Therefore, S has no point of complete accumulation in A. Hence, A is not initially κ-compact.

(4) Let A be a nonclosed subset of X. By (3), there exists a sequence $S = \{x_\alpha\colon \alpha < \lambda\} \subseteq A$, where λ is a (regular) cardinal $\leqslant \kappa$, such that S converges to a nonempty compact set $C \subseteq X \setminus A$. By Lemma 2, there exists $x \in C$ such that $\chi(x, C) \leqslant \kappa$. Let $\mathscr{B}$ be a base of x in C such that $|\mathscr{B}| \leqslant \kappa$. For each $V \in \mathscr{B}$, let $W(V)$ be an open subset of X such that $x \in W(V)$ and $\overline{W(V)} \cap (C \setminus V) = \emptyset$. Let $\mathscr{F}_0$ be a filter of neighbourhoods of x in X such that $\{W(V)\colon V \in \mathscr{B}\} \subseteq \mathscr{F}_0$ and $|\mathscr{F}_0| \leqslant \kappa$. Let $\mathscr{F} = \{F \cap S_\alpha\colon F \in \mathscr{F}_0$ and $\alpha < \kappa\}$, where $S_\alpha = \{x_\beta\colon \beta \geqslant \alpha\}$. Then $\mathscr{F}$ is a filter base of subsets of A, $|\mathscr{F}| \leqslant \kappa$ and $\mathscr{F} \to x$. To prove the last assertion, let U be a neighbourhood of x in X. Let $V \in \mathscr{B}$ be such that $V \subseteq U \cap C$. Since $C \subseteq U \cup (X \setminus \overline{W(V)})$, there exists α such that $S_\alpha \subseteq U \cup (X \setminus \overline{W(V)})$. Then $S_\alpha \cap W(V) \subseteq U$. Also, $S_\alpha \cap W(V) \in \mathscr{F}$. This shows that $\mathscr{F} \to x$.

(5) To prove that $t(X) \leqslant \kappa$, let A be an arbitrary subset of X, and let $Z = \bigcup\{\overline{B}\colon B \subseteq A$ and $|B| \leqslant \kappa\}$. To show that $Z = \overline{A}$, it is enough to show that Z is closed in X. Assume the contrary. Then by (3), there exists $S = \{x_\alpha\colon \alpha < \lambda\} \subseteq Z$, where $\lambda \leqslant \kappa$, and there exists a nonempty compact set $C \subseteq X \setminus Z$ such that $x_\alpha \to C$. For each $\alpha < \lambda$, fix $B_\alpha \subseteq A$ such that $|B_\alpha| \leqslant \kappa$ and $x_\alpha \in \overline{B}_\alpha$. Let $B = \bigcup_{\alpha < \lambda} B_\alpha$. Then $|B| \leqslant \kappa$. Therefore, $\overline{B} \subseteq Z$. Since $S \subseteq \overline{B}$, $C \cap \overline{B} \neq \emptyset$. Hence $C \cap Z \neq \emptyset$. This is a contradiction.

(6) Let D_0 be a dense subset of X such that $|D_0| = d(X)$, and let $Y = \{x \in X\colon \chi(x, X) \leqslant \kappa\}$. Since $\overline{Y} = X$ and $t(X) \leqslant \kappa$, for each $x \in D_0$ there exists $A_x \subseteq Y$ such that $|A_x| \leqslant \kappa$ and $x \in \overline{A}_x$. Let $D = \bigcup\{A_x\colon x \in D_0\}$. Then D is dense in X and $|D| \leqslant d(X) \cdot \kappa$. Since, for each $x \in D$, $\chi(x, X) \leqslant \kappa$, $w(D) \leqslant |D| \cdot \kappa \leqslant d(X) \cdot \kappa$.

(7) Again, let $Y = \{x \in X\colon \chi(x, X) \leqslant \kappa\}$. By a well-known theorem of Hajnal and Juhász (see [9, Theorem 2.15(b)]), $|Y| \leqslant 2^{c(Y) \cdot \kappa)}$. Since Y is dense in X, $c(Y) = c(X)$. Therefore, $|Y| \leqslant 2^{c(X) \cdot \kappa}$.

For each infinite subset A of X, fix a point of complete accumulation $x(A)$ of A in X. Denote by $T(A)$ the set $A \cup \{x(B)\colon B \subseteq A$ and $\omega_0 \leqslant |B| \leqslant \kappa\}$. Then, clearly, $|T(A)| \leqslant |A|^\kappa$.

Let $Z_0 = Y$. Suppose that $\alpha < \kappa^+$ and that Z_β has been defined, for each $\beta < \alpha$.

Let $Z_\alpha = T(\bigcup_{\beta < \alpha} Z_\beta)$. Clearly, $Z_0 \subseteq Z_1 \subseteq \cdots$ and $|Z_\alpha| \leqslant |Y|^\kappa$, for each $\alpha < \kappa^+$. Let $Z = \bigcup_{\alpha < \kappa^+} Z_\alpha$. Then every infinite subset of Z of cardinality $\leqslant \kappa$ has a point of complete accumulation in Z. Therefore, Z is initially κ-compact. Hence, by (2), Z is closed in X. But Z contains the dense subspace Y. Therefore, $Z = X$. Thus, $|X| \leqslant \kappa^+ \cdot |Y|^\kappa \leqslant \kappa^+ \cdot 2^{c(X)\cdot\kappa} = 2^{c(X)\cdot\kappa}$.

(8) In order to prove (8), we first introduce a definition and prove the following lemma.

Definition. If A is a subset of a topological space Z and μ is an infinite cardinal number, then the *μ-closure* of A is the set $(A)_\mu = \{y \in Z$: for each $\lambda < \mu$ and for each G_λ-set U containing y, $U \cap A \neq \emptyset\}$.

Lemma. *Let A be a subset of a topological space Z, and let $A = \bigcup_{\alpha<\kappa} A_\alpha$, where $\kappa < \mu$ and μ is an infinite regular cardinal. If $y \in (A)_\mu$, then there exists $\beta < \kappa$ such that $y \in (A_\beta)_\mu$.*

Proof. Assume the contrary. Then for each $\alpha < \kappa$, there exist $\lambda_\alpha < \mu$ and a G_{λ_α}-set U_α of Z such that $y \in U_\alpha$ and $U_\alpha \cap A_\alpha = \emptyset$. Let $\lambda = \sup\{\lambda_\alpha$: $\alpha < \kappa\}$. Then $\lambda < \mu$ and $U = \bigcap_{\alpha<\kappa} U_\alpha$ is a G_λ-set containing y. But $U \cap A = \emptyset$. This is a contradiction. □

Proof of (8). Let $x \in X$ and $\chi(x, X) = \mu > \kappa$, where μ is a regular cardinal. Since $\psi(x, X) = \mu$, $x \in (X \setminus \{x\})_\mu$. Thus, by the preceding lemma, there exists $\beta < \kappa$ such that $x \in (X_\beta \setminus \{x\})_\mu$. This implies that $x \notin X_\beta$. Indeed, if $x \in X_\beta$, then since $\psi(X_\beta) \leqslant \kappa$, there exists a G_κ-subset U of X such that $U \cap X_\beta = \{x\}$. Then $U \cap (X_\beta \setminus \{x\}) = \emptyset$. This is a contradiction. Thus $x \in (X_\beta)_\mu \setminus X_\beta$.

Now, we can cover X_β by open sets whose closures all miss the point x. Let $\mathscr{H} = \bigcup_{\alpha<\kappa} \mathscr{H}_\alpha$ be a refinement of such a cover (by sets open in X_β) which witnesses the κ-refinability of X_β. For each $\alpha < \kappa$, let $Z_\alpha = \{y \in X_\beta$: $1 \leqslant |\mathscr{H}_\alpha(y)| \leqslant \kappa\}$, where $\mathscr{H}_\alpha(y) = \{V \in \mathscr{H}_\alpha$: $y \in V\}$. Then $X_\beta = \bigcup_{\alpha<\kappa} Z_\alpha$. Therefore, by the preceding lemma, there exists $\gamma < \kappa$ such that $x \in (Z_\gamma)_\mu$.

Now we show that if $\mathscr{G} \subseteq \mathscr{H}_\gamma$ and $|\mathscr{G}| < \mu$, then $x \in (Z_\gamma \setminus \bigcup \mathscr{G})_\mu$. Let $\lambda < \mu$ and let U be a G_λ-subset of X containing x. For each $V \in \mathscr{G}$, let $W(V)$ be an open subset of X such that $x \in W(V)$ and $W(V) \cap V = \emptyset$. Let $P = \bigcap\{W(V)$: $V \in \mathscr{G}\} \cap U$ and $\theta = \max\{|\mathscr{G}|, \lambda\}$. Then $\theta < \mu$ and P is a G_θ-set. Therefore, $\emptyset \neq P \cap Z_\gamma \subseteq Z_\gamma \setminus \bigcup \mathscr{G}$.

Now, let $\{W_\alpha$: $\alpha < \mu\}$ be a base of x in X. By induction, we choose, for each $\alpha < \mu$, $x_\alpha \in Z_\gamma$ as follows. Let $x_0 \in W_0 \cap Z_\gamma$ be arbitrary. If $\alpha < \mu$, and for each $\delta < \alpha$ we have chosen $x_\delta \in Z_\gamma$, then, since $|\bigcup_{\delta<\alpha} \mathscr{H}_\gamma(x_\delta)| < \mu$, by what we showed in the preceding paragraph, it follows that the set $\bigcap_{\delta<\alpha} W_\delta \cap (Z_\gamma \setminus \bigcup\bigcup_{\delta<\alpha} \mathscr{H}_\gamma(x_\delta)) \neq \emptyset$. Choose x_α belonging to this set.

Let $S = \{x_\alpha$: $\alpha < \mu\}$. Then $S \subseteq Z_\gamma \subseteq \bigcup \mathscr{H}_\gamma$, and each member of $\mathscr{H}_\gamma$ contains at most one element of S. Therefore, S is discrete and closed in Z_γ. Clearly, $x_\alpha \to x$. □

Remarks 7. (a) If a (discrete) sequence of length μ converges to a point x in a space X, where μ is an infinite regular cardinal, then $\chi(x, X) \geqslant \mu$. Therefore, if X is a space as in Theorem 6 and $x \in X$, then, in view of (8), $\chi(x, X) > \kappa$ if and only if there is a discrete sequence of length $> \kappa$ in X converging to x. Also, from (8) it follows that, if $s(X) \leqslant \kappa$, then $\chi(X) \leqslant \kappa$. ($s(X)$ denotes the spread of X.)

(b) From (3), it follows that if x is a nonisolated point of X, then there exists a sequence $\{x_\alpha: \alpha < \lambda\}$ in X, where λ is a regular cardinal $\leqslant \kappa$, such that $x_\alpha \to x$.

(c) If $\kappa \leqslant \omega_0$, then it follows from (4) that for every nonclosed subset A of X, there exist a point $x \in X \setminus A$ and a sequence $\{x_n\}$ in A such that $x_n \to x$, i.e., the space X is sequential. Thus the main theorem of [11], namely, σ-metric compact Hausdorff spaces are sequential, follows from (4).

(d) (4) can be restated as "$\operatorname{div}(X) \leqslant \kappa$". Divergence of a topological space X, denoted by $\operatorname{div}(X)$, is defined to be the smallest cardinal number τ such that, if A is a nonclosed subset of X, then there exist a point $x \in X \setminus A$ and a filter base $\mathscr{F}$ of subsets of A such that $|\mathscr{F}| \leqslant \tau$ and $\mathscr{F} \to x$ (see [1]).

(e) In [3], Arhangel'skĭi asked the following questions: let X be a compact Hausdorff space, and let $X = \bigcup_{\alpha<\kappa} X_\alpha$, where each X_α is metrizable, $\kappa > \omega_0$ and $c(X) \leqslant \kappa$. Is it true then that $d(X) \leqslant \kappa$ or that $\pi w(X) \leqslant \kappa$? Must X then contain a dense subspace Y such that $w(Y) \leqslant \kappa$?

In view of (6), if $d(X) \leqslant \kappa$, then X does contain a dense subspace D such that $w(D) \leqslant \kappa$. This answers the second question in affirmative. The first question, however, remains open (see Problem 20 below). When $\kappa \leqslant \omega_0$, both the questions have affirmative answers (see Corollary 13 below). When the cardinal κ satisfies a strong inaccessibility type condition, then we have the following partial answer to the above questions.

Proposition 8. *Let κ be an unaccountable regular cardinal such that $2^\lambda \leqslant \kappa$, for each $\lambda < \kappa$. Let X be a regular initially κ-compact space, and let $X = \bigcup_{\alpha<\kappa} X_\alpha$, where $\psi(X_\alpha) < \kappa$, for each $\alpha < \kappa$. Let $c(X) < \kappa$. Then there exists a dense subspace Y of X such that $w(Y) \leqslant \kappa$.*

Proof. Let $Y = \{x \in X\colon \chi(x, X) < \kappa\}$. Then, by Lemma 2(b), $\overline{Y} = X$. Also, $Y = \bigcup\{Y(\lambda)\colon \lambda < \kappa\}$, where $Y(\lambda) = \{x \in X\colon \chi(x, X) \leqslant \lambda\}$. By [9, Theorem 2.15(b)], $|Y(\lambda)| \leqslant 2^{c(X)\cdot\lambda}$, for each λ. By the given condition on κ, $2^{c(X)\cdot\lambda} \leqslant \kappa$. Therefore, $|Y(\lambda)| \leqslant \kappa$, fore each λ. Hence $|Y| \leqslant \kappa$. Since $\chi(Y) \leqslant \kappa$, $w(Y) \leqslant \kappa$. $\square$

Suppose that a space X satisfies the hypothesis of Theorem 6, $x \in X$ and $\chi(x, X) = \mu > \kappa$, where μ is a regular cardinal. By (8), we know that there exists a discrete sequence of length μ in X which converges to the point x. We now consider the following question: Does there exist a discrete sequence S of length μ in X such that $S \to x$ and x is the only limit point of S?

Definition 9. A space Z is called *strongly κ-refinable* if every open cover of Z has an open refinement $\mathscr{H}$ such that, for each $z \in Z$, $1 \leqslant |\{V \in \mathscr{H}\colon z \in V\}| \leqslant \kappa$.

Suppose that X is as above and each X_α is strongly κ-refinable. Then, as in the proof of (8), above, there exists $\beta_0 < \kappa$ such that $x \in (X_{\beta_0})_\mu$. Let $\mathscr{H}$ be a cover of X_{β_0} by sets open in X_{β_0} such that $x \notin \overline{V}$, for each $V \in \mathscr{H}$, and for each $y \in X_{\beta_0}$, $|\{V \in \mathscr{H}: y \in V\}| \leqslant \kappa$. Then, as in the proof of (8), by replacing Z_γ by X_{β_0} and $\mathscr{H}_\gamma$ by $\mathscr{H}$, we can construct a sequence S_0 of length μ contained in X_{β_0} such that S_0 is discrete and closed in X_{β_0} and $S_0 \to x$. Clearly, $(\overline{S}_0 \setminus S_0) \cap X_{\beta_0} = \emptyset$.

The compact Hausdorff space $\overline{S}_0 \setminus S_0$ is of the same type as the space X and $x \in \overline{S}_0 \setminus S_0$. If $\chi(x, \overline{S}_0 \setminus S_0) = \mu$, then we can repeat the above argument, with $\overline{S}_0 \setminus S_0$ in place of X, and find a $\beta_1 (\neq \beta_0) < \kappa$ and a discrete sequence S_1 of length μ contained in $\overline{S}_0 \setminus S_0$ such that $S_1 \to x$ and $(\overline{S}_1 \setminus S_1) \cap X_{\beta_1} = \emptyset$.

Suppose, now, that $X = \bigcup_{\alpha < n} X_\alpha$, where n is finite and $X_\alpha s$ satisfy the same properties as above. Then, clearly, the above process must end after finitely many steps, and we will get a discrete sequence $S = \{x_\alpha: \alpha < \mu\} \subseteq X$ such that $S \to x$ and $\chi(x, \overline{S} \setminus S) = \lambda < \mu$. Let F be a closed G_λ-subset of X such that $F \cap (\overline{S} \setminus S) = \{x\}$. There exists $\beta < \mu$ such that $S_\beta = \{x_\alpha: \alpha \geqslant \beta\} \subseteq F$. Then $\overline{S}_\beta \setminus S_\beta = \{x\}$. Therefore, $S_\beta \cup \{x\}$ is the one-point compactification of the discrete space S_β. We have, thus, proved the following theorem.

Theorem 10. *Let X be a compact Hausdorff space, and let $X = \bigcup_{\alpha < n} X_\alpha$, where n is finite, each X_α is strongly κ-refinable and $\psi(X_\alpha) \leqslant \kappa$. If $x \in X$ and $\chi(x, X) = \mu > \kappa$, where μ is a regular cardinal then there exists a discrete sequence S of length μ in X such that x is the only limit point of S in X.*

Whether this theorem can be proved without the condition "n is finite" remains open (see Problem 19 below).

2.

We now consider the case when a compact Hausdorff space is represented as a union of countable many metrizable spaces. Typical nonmetrizable examples of such spaces would be the one-point compactification of an uncountable discrete space, which is a union of two metrizable spaces, or the one-point compactification of the space Ψ (see [5, 5I]), which is a union of three metrizable spaces, or the Alexandroff duplicate of the unit segment, which is a union of two metrizable spaces.

Theorem 11. *Let X be a compact Hausdorff space, and let $X = \bigcup_{i=1}^\infty X_i$, where each X_i is metrizable. Then X contains a dense G_δ-metrizable subspace.*

Proof. We first show that there exists a dense G_δ-subset G of X such that $G \subseteq \bigcup_{i=1}^\infty Z_i$, where each Z_i is metrizable and dense in an open subset V_i of X.

For each i, let $U_i = \mathrm{Int}(X \setminus X_i)$. Then $X_i \cap \overline{U}_i = X_i \cap (\overline{U}_i \setminus U_i)$ is nowhere dense in X. Hence by the Baire Category Theorem, there exists a dense G_δ-subset G of

X such that $G \cap (X_i \cap \overline{U}_i) = \emptyset$, for each i. Let $Z_i = X_i \setminus \overline{U}_i$ and $V_i = X \setminus \overline{U}_i$. Then $G \subseteq \bigcup_{i=1}^{\infty} Z_i$ and Z_i is dense in V_i, for each i.

We now prove the following lemma.

Lemma. *Let Y be a regular space, and let $Z \subseteq V \subseteq Y$, where V is open in Y, Z is metrizable and Z is dense in V. Then there exists a collection $\mathscr{B} = \bigcup_{i=1}^{\infty} \mathscr{B}_i$ of open subsets of V, and there exists a sequence $W_1, W_2, \ldots$ of open dense subsets of Y such that $Z \subseteq W_i$, for each i, and the following two conditions are satisfied.*

(a) *For each i, $\mathscr{B}_i$ is locally finite in Y at each point of W_i.*

(b) *$\mathscr{B}$ is a base (in Y) at each point of Z.*

Proof. Let $\mathscr{S} = \bigcup_{i=1}^{\infty} \mathscr{S}_i$ be a base of Z, where each $\mathscr{S}_i$ is locally finite in Z. For each $S \in \mathscr{S}$, let $U(S)$ be an open subset of Y such that $U(S) \subseteq V$ and $U(S) \cap Z = S$. For each i, let $\mathscr{B}_i = \{U(S)\colon S \in \mathscr{S}_i\}$, and let $\mathscr{B} = \bigcup_{i=1}^{\infty} \mathscr{B}_i$. Also, for i, let $W_i = \{y \in Y\colon \mathscr{B}_i$ is locally finite at y in $Y\}$. Since Z is dense in V, $Z \subseteq W_i$. Also, $Y \setminus \overline{V} \subseteq W_i$. Therefore, W_i is dense in Y. Clearly, W_i is open in Y.

Condition (a) is obviously satisfied. To verify (b), let $y \in Z$, and let U be an open subset of Y such that $y \in U$. Choose an open subset U' of Y such that $y \in U' \subseteq \overline{U}' \subseteq U \cap V$. There exists $S \in \mathscr{S}$ such that $y \in S \subseteq U' \cap Z$. Then $U(S) \in \mathscr{B}$ and $y \in U(S) \subseteq \overline{S} \subseteq \overline{U}' \subseteq U$. Hence $\mathscr{B}$ is a base (in Y) at each point of Z. The proof of the lemma is complete. □

Proof of Theorem 11 (*continued*). By the lemma, for each $i = 1, 2, 3 \ldots$, let $\mathscr{B}(i) = \bigcup_{j=1}^{\infty} \mathscr{B}(i, j)$ be a collection of open subsets of V_i, and let $W(i, 1), W(i, 2), \ldots$, be a sequence of open dense subsets of X each containing Z_i such that $\mathscr{B}(i)$ is a base (in X) at each point of Z_i and for each j, $\mathscr{B}(i, j)$ is locally finite in X at each point of $W(i, j)$. Let $H = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} W(i, j) \cap G$. By the Baire Category Theorem, H is dense in X. Also, H is G_δ in X. Furthermore, $\mathscr{B} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathscr{B}(i, j)$ is a base (in X) at each point of H, and for each i and, for each j, $\mathscr{B}(i, j)$ is locally finite (in X) at each point of H. Hence, H is metrizable. □

Clearly, in the above proof, instead of assuming that X is a compact Hausdorff space, we only needed to assume that X is a regular Baire space.

Corollary 12. *Let X be a compact Hausdorff space, and let $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is metrizable. If X has no isolated points and $c(X) = \omega_0$, the X contains a dense G_δ-subspace homeomorphic to the space of irrationals.*

Proof. By the above theorem, X contains a dense G_δ-metrizable subspace H. Then H is completely metrizable, and since $c(H) = c(X) = \omega_0$, H is separable. Also, H has no isolated points. Such a space H contains a dense G_δ-subspace homeomorphic to the space of irrationals (see [10, Theorem 3, p. 441]). □

If X is a regular space, and H is a dense metrizable subspace of X, then $c(H) = d(H) = \pi w(H) = w(H) = c(D) = \pi w(D) = d(D)$, where D is an arbitrary dense subspace of X (see [9]). Thus, we have the following corollary.

Corollary 13 [2, Theorem I.18]. *Let X be a compact Hausdorff space, and let $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is metrizable. Then $c(X) = d(X) = \pi w(X) = d(D)$, where D is an arbitrary dense subspace of X. Also, there exists a dense (metrizable) subspace H of X such that $w(H) = c(X)$.*

We now state two generalizations of the above theorem. The first can be proved in exactly the same way as the above theorem. An outline of the proof of the second is given.

A collection $\mathscr{G}$ of subsets of a space Y is called *locally-κ* if for each $y \in Y$, there exists a neighborhood U of y such that $|\{V \in \mathscr{G}: U \cap V \neq \emptyset\}| \leqslant \kappa$.

Theorem 14. *Let X be a compact Hausdorff space, and let $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i has a σ-locally-κ base. Then X contains a dense G_δ-subspace which has a σ-locally-κ base.*

A space is called a σ-space if it has a σ-discrete network (cf. [6]).

Theorem 15. *Let X be a compact Hausdorff space, and let $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is a σ-space. Then X contains a dense G_δ-metrizable subspace.*

Proof. By an argument similar to (and shorter than) the proof of Theorem 11, we first show that X contains a dense G_δ-subspace which is a σ-space. We outline the argument.

As in the proof of Theorem 11, there exists a dense G_δ-subspace G of X such that $G \subseteq \bigcup_{i=1}^{\infty} Z_i$, where for each i, Z_i is a σ-space and Z_i is dense in an open subset V_i of X. For each i, let $\mathscr{G}(i) = \bigcup_{j=1}^{\infty} \mathscr{G}(i, j)$ be a network of Z_i, where, for each j, $\mathscr{G}(i, j)$ is discrete in Z_i. For each i, and for each j, let $W(i, j) = \{x \in X: \mathscr{G}(i, j)$ is discrete at x in $X\}$. Then $Z_i \subseteq W(i, j)$, and $W(i, j)$ is open and dense in X. Let $H = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} W(i, j) \cap G$. Then H is a dense G_δ-subset of X. Also, $\mathscr{G} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathscr{G}(i, j)$ is a σ-discrete network of H in X. Hence H is a σ-space.

The space H, being both a σ-space and a Čech-complete space, is developable (see [6, Corollary 4.7]). The rest of the proof follows from the following known lemma (see [7, Theorem 5.1]).

Lemma. *A completely regular Čech-complete developable space contains a dense G_δ-metrizable subspace.* □

Remark 16. In view of the above theorem, Corollaries 12 and 13 remain true if the phrase "X_i is metrizable" is replaced by the phrase "X_i is a σ-space".

Theorem 17. *Let X be a compact Hausdorff space, and let $X = \bigcup_{i=1}^{\infty} X_i$, where for each i, $\chi(X_i) \leqslant \kappa$. Then X contains a dense G_δ-subspace G such that $\chi(G) \leqslant \kappa$.*

Proof. As in the proof of Theorem 11, there exists a dense G_δ-subspace G of X such that $G \subseteq \bigcup_{i=1}^{\infty} Z_i$, where, for each i, $\chi(Z_i) \leqslant \kappa$ and Z_i is dense in an open subset V_i of X.

Let $x \in G$. Then for some i, $x \in Z_i$. Since Z_i is dense in V_i, $\chi(x, V_i) = \chi(x, Z_i) \leqslant \kappa$. But V_i is open in X. Therefore, $\chi(x, X) \leqslant \kappa$. Hence $\chi(x, G) \leqslant \kappa$. Therefore, $\chi(G) \leqslant \kappa$. $\square$

In the above proof, instead of assuming that X is a compact Hausdorff space, we only needed to assume that X is a regular Baire space.

Finally, we would like to state the following problems which we have not been able to resolve.

Let X be compact Hausdorff space satisfying the hypothesis of Theorem 6.

Problem 18. Does the set $Y = \{x \in X\colon \chi(x, X) \leqslant \kappa\}$ contain an open dense subset of X?

Problem 19. If $x \in X$, $\chi(x, X) = \mu > \kappa$, where μ is a regular cardinal, then does there exist a discrete sequence S of length μ in X such that x is the only limit point of S? (By Theorem 6(8), we know that there exists a discrete sequence S of length μ in X such that $S \to x$). What if μ is a singular cardinal?

Note. Both problems remain unresolved even if each X_α is assumed to be metrizable.

We would also like to state the aforementioned problem of Arhangel'skĭi (see Remark 7(e)):

Problem 20. Let X be a compact Hausdorff space, and let $X = \bigcup_{\alpha < \kappa} X_\alpha$, where each X_α is metrizable and $\kappa > \omega_0$. Is it true that $d(X) \leqslant c(X) \cdot \kappa$?

Note added in proof

The authors have found an example which answers Problem 18 in the negative.

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